On Steinhaus properties and families of "small" sets

Eliza Jabłońska
(joint work with Taras Banakh, Szymon Głąb, Jarosław Swaczyna)

Department of Discrete Mathematics
Faculty of Mathematics and Applied Physics
Rzeszów University of Technology

Hejnice, Jan 27 - Feb 3, 2018

All groups considered here are Abelian.

A family $\mathcal{F}$ of subsets of a Polish group X is defined to be

- *Steinhaus-Smital* if for any Borel subsets $A, B \notin \mathcal{F}$ the difference $A - B$ has non-empty interior in X;
- *weak Steinhaus-Smital* if for any Borel subsets $A, B \notin \mathcal{F}$ the difference $A - B$ is non-meager in X;
- *Steinhaus* if for any Borel subset $A \notin \mathcal{F}$ the difference $A - A$ is a neighborhood of zero in X;
- *weak Steinhaus* if for any Borel subset $A \notin \mathcal{F}$ the difference $A - A$ is non-meager in X.

It is clear that

$$\begin{array}{ccc} \text{Steinhaus-Smital} & \Rightarrow & \text{weak Steinhaus-Smital} \\ & & \Downarrow \\ \text{Steinhaus} & \Rightarrow & \text{weak Steinhaus} \end{array}$$

Steinhaus-Smital $\not\Rightarrow$ Steinhaus

Example 1 (Banakh-Głąb-J.-Swaczyna)

Let $\mathcal{F}$ be the semi-ideal of all subsets $A \subset \mathbb{C}$ such that for any unit circle $S(z_0; 1)$ the intersection $A \cap S(z_0; 1)$ has empty interior in $S(z_0; 1)$. Then $\mathcal{F}$ is Steinhaus-Smital but not Steinhaus in $\mathbb{C}$.

Problem 1

Is there an ideal $\mathcal{I}$ of subsets of a Polish group such that $\mathcal{I}$ is Steinhaus-Smital but not weak Steinhaus?

$\mathcal{M}$ – the σ-ideal of meager sets in a topological space X.
$\mathcal{N}$ – the σ-ideal of sets of Haar measure zero in a locally compact topological group X.
$\sigma\overline{\mathcal{N}}$ – the σ-ideal generated by closed sets of Haar measure zero in a locally compact topological group X.

Steinhaus Theorem (1920)

For every locally compact Polish group the ideal $\mathcal{N}$ is Steinhaus-Smital and Steinhaus.

- H. Steinhaus, *Sur les distances des points des ensembles de mesure positive*, Fund. Math. 1 (1920), 99-104.

Pettis–Piccard Theorem (1939, 1951)

For every Polish group the ideal $\mathcal{M}$ is Steinhaus-Smital and Steinhaus.

- S. Piccard, *Sur les ensembles de distances des ensembles de points d'un espace Euclidien*, Mem. Univ. Neuchâtel, vol. 13, Secrétariat Univ., Neuchâtel, 1939.
- B.J. Pettis, *Remarks on a theorem of E. J. McShane*, Proc. Amer. Math. Soc. 2 (1951), 166-171.

Corollary

For every locally compact Polish group the ideal $\mathcal{M} \cap \mathcal{N}$ is Steinhaus.

Example (Bartoszewicz-Filipczak-Natkaniec, 2011)

For every locally compact Polish group the ideal $\mathcal{M} \cap \mathcal{N}$ is not Steinhaus-Smital.

- A. Bartoszewicz, M. Filipczak, T. Natkaniec, *On Smital properties*, Topology Appl. 158 (2011), 2066-2075.

Theorem (Banakh-Głąb-J.-Swaczyna)

For every locally compact Polish group the ideal $\sigma\overline{\mathcal{N}} \subset \mathcal{M} \cap \mathcal{N}$ is weak Steinhaus-Smital.

Example (Banakh-Głąb-J.-Swaczyna)

For the compact metrizable topological group $X = \prod_{n\in\omega} C_{2^n}$, where $C_{2^n} := \mathbb{Z}/2^n\mathbb{Z}$, the ideal $\sigma\overline{\mathcal{N}}$ is neither Steinhaus-Smital nor Steinhaus. (A is a G_δ-subset of the set $\prod_{n\in\omega} C^*_{2^n}$)

For every locally compact Polish group:

Ideal	S-S	S	weak S-S	weak S
$\mathcal{M} = \sigma\overline{\mathcal{M}}$	Y	Y	Y	Y
$\mathcal{N}$	Y	Y	Y	Y
$\mathcal{M} \cap \mathcal{N}$	N	Y	Y	Y
$\sigma\overline{\mathcal{N}}$	N	N	Y	Y

Definition 1 (Christensen, 1972)

A subset A of a Polish group X is called *Haar null*, if there are a Borel set $B \subset X$ containing A and a probability σ-additive Borel measure μ on X such that $\mu(x + B) = 0$ for each $x \in X$.

- The family $\mathcal{HN}$ of all Haar null sets is a σ-ideal.
- If X is locally compact, then $\mathcal{HN} = \mathcal{N}$.

Theorem (Christensen, 1972, Matoušková-Zelený, 2003)

In every Polish group the ideal $\mathcal{HN}$ is Steinhaus.
In any non-locally compact Polish group $\mathcal{HN}$ is not Steinhaus-Smital.

- J.P.R. Christensen, *On sets of Haar measure zero in abelian Polish groups*, Israel J. Math. 13 (1972), 255-260.
- E. Matoušková, M. Zelený, *A note on intersections of non–Haar null sets*, Colloq. Math. 96 (2003), 1-4.

Definition 2 (Darji, 2013)

Let X be a Polish group. A set $A \subset X$ is called *Haar meager*, if there exists a Borel set $B \subset X$ containing A, a compact metric space K and a continuous function $f : K \to X$ such that $f^{-1}(B + x)$ is meager in K for each $x \in X$.

- The family $\mathcal{HM}$ of all Haar meager sets is a σ-ideal.
- If X is locally compact, then $\mathcal{HM} = \mathcal{M}$.
- If X is not locally compact, then $\mathcal{HM} \subsetneq \mathcal{M}$.

- U.B. Darji, *On Haar meager sets*, Topology Appl. 160 (2013), 2396-2400.

Theorem (J., 2015)

In every Polish group the ideal $\mathcal{HM}$ is Steinhaus.
In any non-locally compact Polish group $\mathcal{HM}$ is not Steinhaus-Smital.

- E. Jabłońska, *Some analogies between Haar meager sets and Haar null sets in abelian Polish groups*, J. Math. Anal. Appl. 421 (2015), 1479-1486.

Example (folklore)

For the group $X = \mathbb{Z}^\omega$ the ideals $\mathcal{HN}$ and $\mathcal{HM}$ are not weak Steinhaus-Smital. ($A = \omega^\omega$)

In a locally compact Polish group:

Ideal	S-S	S	weak S-S	weak S
$\mathcal{M} = \mathcal{HM}$	Y	Y	Y	Y
$\mathcal{N} = \mathcal{HN}$	Y	Y	Y	Y

In a non-locally compact Polish group:

Ideal	S-S	S	weak S-S	weak S
$\mathcal{M}$	Y	Y	Y	Y
$\mathcal{HM}$	N	Y	N	Y
$\mathcal{HN}$	N	Y	N	Y

Theorem (Banakh-Głąb-J.-Swaczyna)

Each closed Haar null subset of a Polish group is Haar meager. Consequently $\overline{\mathcal{HN}} \subset \overline{\mathcal{HM}}$ and $\sigma\overline{\mathcal{HN}} \subset \sigma\overline{\mathcal{HM}}$.

Example 2 (Banakh-Głąb-J.-Swaczyna)

The Polish group $\mathbb{Z}^\omega$ contains a meager σ-Polish subgroup Y, which does not belong to the ideal $\sigma\overline{\mathcal{HM}}$ in $\mathbb{Z}^\omega$, Consequently, $\sigma\overline{\mathcal{HM}}$ is not weak Steinhaus.

In a locally compact Polish group:

Ideal	S-S	S	weak S-S	weak S
$\sigma\overline{\mathcal{HN}} = \sigma\overline{\mathcal{N}}$	N	N	Y	Y
$\sigma\overline{\mathcal{HM}} = \sigma\overline{\mathcal{M}} = \mathcal{M}$	Y	Y	Y	Y

In a non-locally compact Polish group:

Ideal	S-S	S	weak S-S	weak S
$\sigma\overline{\mathcal{HN}}$	N	N	N	N
$\sigma\overline{\mathcal{HM}}$	N	N	N	N

Definition 3 (Dodos, 2004)

Let $\mathcal{P}(X)$ be the space of all probability σ-additive Borel measures on a Polish group X. For a Borel set $A \subset X$

$$T(A) := \{\mu \in \mathcal{P}(X) : \ \mu(x + A) = 0 \ \text{ for all } \ x \in X\}.$$

Theorem (Dodos, 2004)

If A is a Borel Haar null subset of a Polish group X, then $T(A)$ is either meager or comeager in $\mathcal{P}(X)$.

- P. Dodos, *Dichotomies of the set of test measures of a Haar null set*, Israel J. Math. 144 (2004), 15-28.
- P. Dodos, *On certain regularity properties of Haar-null sets*, Fund. Math. 181 (2004), 97-109.

Definition 4 (Dodos, 2004)

A subset A of a Polish group X is called *generically Haar null* provided $T(A)$ is comeager in $\mathcal{P}(X)$.

$\mathcal{GHN}$ – the σ-ideal generated by Borel generically Haar null sets in X.

Theorem (Dodos, 2009)

The ideal $\mathcal{GHN}$ in a Polish group X is weak Steinhaus.

- P. Dodos, *The Steinhaus property and Haar-null sets*, Bull. Lond. Math. Soc. 41 (2009), 377–384.

Proposition (Banakh-Głąb-J.-Swaczyna)

Let $A \subset X$ be a Borel set. The following conditions are equivalent:

(i) A is Haar meager;

(ii) there exists a continuous function $f : 2^\omega \to X$ such that the set $f^{-1}(A+x)$ is meager in 2^ω for $x \in X$.

Definition 5 (Banakh-Głąb-J.-Swaczyna)

Let $C(2^\omega, X)$ be the space of all continuous functions $f : 2^\omega \to X$, where X is a Polish group. For a Borel set $A \subset X$

$$W(A) := \{f \in C(2^\omega, X) : \ f^{-1}(x + A) \text{ is meager in } 2^\omega \text{ for all } x \in X\}.$$

Theorem (Banakh-Głąb-J.-Swaczyna)

If A is a Borel Haar meager subset of a Polish group X, then $W(A)$ is either meager or comeager in $C(2^\omega, X)$.

Definition 6 (Banakh-Głąb-J.-Swaczyna)

A subset A of a Polish group X is called *generically Haar meager* provided $W(A)$ is comeager in $C(2^\omega, X)$.

$\mathcal{GHM}$ – the σ-ideal generated by Borel generically Haar meager sets in X.

Theorem (Banakh-Karchevska-Ravsky, 2015)

The ideal $\mathcal{GHM}$ in a Polish group X is weak Steinhaus.

- T. Banakh, L. Karchevska, A. Ravsky, *The closed Steinhaus properties of σ-ideals on topological groups*, arXiv:1509.09073v1 [math.GN] 30 Sep 2015.

In a locally compact Polish group:

$$\sigma\overline{\mathcal{N}} \subset \mathcal{GHN} \subset \mathcal{M} \cap \mathcal{N}$$

Semi-ideal	S-S	S	weak S-S	weak S
$\mathcal{GHM} = \mathcal{M}$	Y	Y	Y	Y
$\mathcal{M} \cap \mathcal{N}$	N	Y	Y	Y
$\mathcal{GHN}$	N	?	Y	Y
$\sigma\overline{\mathcal{N}}$	N	N	Y	Y

In a non-locally compact Polish group:

$$\mathcal{GHN} \subset \mathcal{HN}, \quad \mathcal{GHM} \subset \mathcal{HM}$$

Semi-ideal	S-S	S	weak S-S	weak S
$\mathcal{HM}$	N	Y	N	Y
$\mathcal{HN}$	N	Y	N	Y
$\mathcal{GHM}$	N	?	N	Y
$\mathcal{GHN}$	N	?	N	Y

In every **locally compact** Polish group:

$$\sigma\overline{\mathcal{N}} = \sigma\overline{\mathcal{HN}} \subset \mathcal{GHN} \subset \mathcal{M} \cap \mathcal{N} \subset \mathcal{HN} = \mathcal{N},$$

$$\sigma\overline{\mathcal{M}} = \sigma\overline{\mathcal{HM}} = \mathcal{GHM} = \mathcal{HM} = \mathcal{M}.$$

Semi-ideal	S-S	S	weak S-S	weak S
$\mathcal{M} = \mathcal{HM} = \mathcal{GHM} = \sigma\overline{\mathcal{M}}$	Y	Y	Y	Y
$\mathcal{N} = \mathcal{HN}$	Y	Y	Y	Y
$\mathcal{M} \cap \mathcal{N}$	N	Y	Y	Y
$\mathcal{GHN}$	N	?	Y	Y
$\sigma\overline{\mathcal{N}}$	N	N	Y	Y

In every **non-locally compact** Polish group:

$$\sigma\overline{\mathcal{HN}} \subset \mathcal{GHN} \subset \mathcal{M} \cap \mathcal{HN} \subset \mathcal{HN},$$

$$\sigma\overline{\mathcal{HM}} \subset \mathcal{GHM} \subset \mathcal{HM} \subset \mathcal{M}.$$

Semi-ideal	S-S	S	weak S-S	weak S
$\mathcal{M} = \sigma\overline{\mathcal{M}}$	Y	Y	Y	Y
$\mathcal{HN}$	N	Y	N	Y
$\mathcal{HM}$	N	Y	N	Y
$\mathcal{GHM}$	N	?	N	Y
$\mathcal{GHN}$	N	?	N	Y
$\sigma\overline{\mathcal{HM}}$	N	N	N	N
$\sigma\overline{\mathcal{HN}}$	N	N	N	N

Semi-ideal	S-S		S		weak S-S		weak S	
	l.c.	n.l.c.	l.c.	n.l.c.	l.c.	n.l.c.	l.c.	n.l.c.
$\mathcal{M}$	Y	Y	Y	Y	Y	Y	Y	Y
$\mathcal{HN}$	Y	N	Y	Y	Y	N	Y	Y
$\mathcal{HM}$	Y	N	Y	Y	Y	N	Y	Y
$\mathcal{GHM}$	Y	N	Y	?	Y	N	Y	Y
$\mathcal{GHN}$	N	N	?	?	Y	N	Y	Y
$\sigma\overline{\mathcal{HN}}$	N	N	N	N	Y	N	Y	N
$\sigma\overline{\mathcal{HM}}$	Y	N	Y	N	Y	N	Y	N